CFT & gravity: week 7

Correlation functions from embedding formalism

$$G_{ABCD}(P_1, P_2) = \langle \mathcal{O}_{AB}(P_1)\, \mathcal{O}_{CD}(P_2) \rangle$$

has the following properties:

1) $G_{ABCD} = G_{(AB)(CD)}$ $\qquad (\ldots) =$ symmetrization

2) $G^{A}{}_{ACD} = G_{ABC}{}^{C} = 0$

3) $P_1^A G_{ABCD} = P_2^C G_{ABCD} = 0$

4) $G_{ABCD}(\lambda_1 P_1, \lambda_2 P_2) = \lambda_1^{-\Delta} \lambda_2^{-\Delta} G_{ABCD}(P_1, P_2)$

- All properties must be valid on the light-cone, so up to terms proportional to P_1^2, P_2^2.

- The crucial ones is transversality. We want combinations of $\{P_{1A}, P_{2B}, \eta_{AB}\}$ transverse. We start with 2 indices, then we get to 4:

$$W_{AB} = \eta_{AB} + c_1 P_{1A} P_{1B} + c_2 P_{2A} P_{2B} + c_3 P_{1A} P_{2B} + c_4 P_{2A} P_{1B}$$

W should be dimensionless in both P_1 and P_2. But the only Lorentz invariant which does not vanish on the lightcone is $P_1 \cdot P_2$, so we must choose:

- $c_1 = c_2 = 0$
- $c_3 = \alpha / P_1 \cdot P_2 \qquad c_4 = \beta / P_1 \cdot P_2$

$$W_{AB} = \eta_{AB} + \frac{\alpha P_{1A} P_{2B} + \beta P_{2A} P_{1B}}{P_1 \cdot P_2}$$

In principle we could build various W_{AB}'s, transverse wrt P_1 or P_2, symmetric or without symmetry properties etc.

However, let us pause and count how many structures we need. We can use a conformal transformation to send $x_1 \to 0$ and $x_2 \to \infty$. Then the correlator becomes an overlap of states:

$\langle AB | CD \rangle$ which should be invariant under rotations, and traceless and symm. in AB, CD.

there is only one possibility:

$= \delta_{A(C}\delta_{BD)}$ - traces.

So only 1 structure. This is a general fact:
the tensor product of an IRREP with its conjugate contains the identity only once.
So all two-point functions have only one structure.

Therefore we only need to find one W_{AB} that works.

If we had a W_{AB} symmetric and transverse wrt both P_1 and P_2, then we could build:

$$W_{AB} W_{CD} + \alpha W_{A(C} W_{BD)}$$

And fix α by the tracelessness condition. For the latter to work, we also need:

$$W_{AC} W^{A}{}_{D} \propto W_{CD}$$

But this is obeyed by a projector with proportionality coefficient 1.

Conclusion: $\boxed{\text{We look for the projector onto the space orthogonal to } P_1 \text{ and } P_2}$

- Start with transversality on the symmetric W_{AB}:

$$W_{AB} = \eta_{AB} + \gamma \frac{P_{1A}P_{2B} + P_{2A}P_{1B}}{P_1 \cdot P_2}$$

$P_1{}^A W_{AB} = 0 \longrightarrow \boxed{\gamma = -1}$ then automatically $P_2{}^A W_{AB} = 0$

- Then the involution:

$$W_{AB} W^B{}_C = W_{AC} - W_{AB}\left(P_1^B \dots + P_2^B \dots\right) \overset{\checkmark}{=} W_{AC}$$

CS271 F

- The trace: $W_A{}^A = d+2 - 2 = d$

So: $W_{AB}W_{CD} - d\, W_{A(C}W_{B\,D)}$

$$M_{(AB)} = \frac{1}{2}\left(M_{AB} + M_{BA}\right)$$

Obeys 1) to 3). To get 4) we only have to rescale with $P_1 \cdot P_2$:

$$\langle \mathcal{O}_{AB}(P_1)\,\mathcal{O}_{CD}(P_2)\rangle = C\,\frac{W_{AB}W_{CD} - d\,W_{A(C}W_{B\,D)}}{(-2P_1\cdot P_2)^{\Delta}}$$

$$W_{AB} = \eta_{AB} - \frac{P_{1A}P_{2B} + P_{2A}P_{1B}}{P_1\cdot P_2}$$

(Observe: parts $\propto P_{1A}, P_{1B}, P_{2C}, P_{2D}$ are pure gauge and can be dropped)

-2 just for convenience when projecting on the Poincaré section.

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Conformal blocks from Casimir differential equation

a) see Mathematica notebook

b) $u = z\bar{z} = \dfrac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2}$ $\qquad v = (1-z)(1-\bar{z}) = \dfrac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2}$

$x_4 \to \infty \qquad x_{13}^2 = 1$

$\rightsquigarrow \quad z\bar{z} = x_{12}^2 \qquad (1-z)(1-\bar{z}) = x_{23}^2$

- $z\bar{z} = |z|^2 = x_{12}^2$

- $x_{23}^2 = (x_{21} + x_{13})^2 = x_{12}^2 + 1 + 2\, x_{12}\cdot x_{13}$

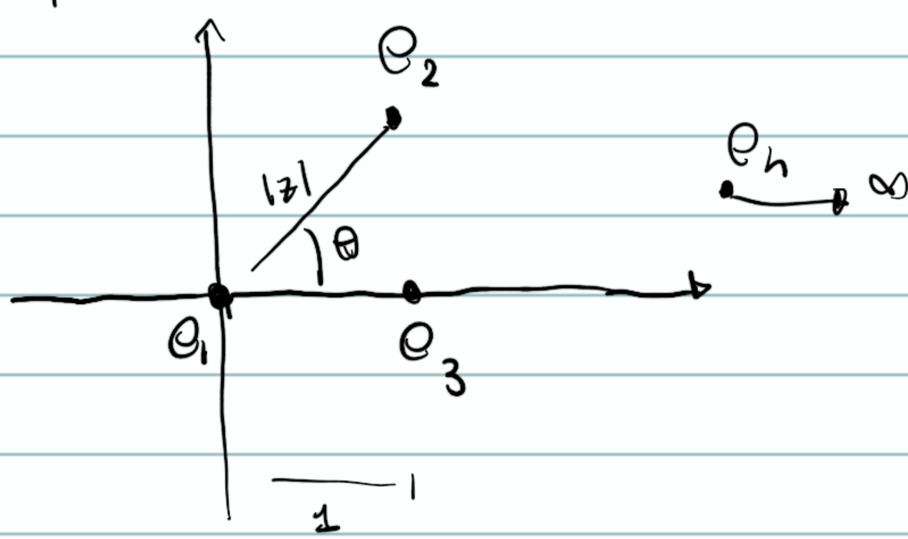
$$= |z|^2 + 1 + \;'' \qquad = (1-z)(1-\bar{z})$$

$\longrightarrow$ $x_{12}\cdot x_{13} = \mathrm{Re}\, z$

$|z| \cos\theta_{\widehat{12\,13}} = \mathrm{Re}\, z$

So: $z = |x_{12}|\, e^{i\theta_{\widehat{12\,13}}} = |x_{12}|\, e^{i\theta}$

Here is the picture:



c), d), e) See Mathematica notebook